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Damage spreading in the Potts and Ashkin–Teller models: exact results

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Abstract. Following the approach introduced by Coniglio *et al*, for the Ising model, we study in this paper the relation between damage spreading and thermal properties in the q-state Potts model as well as in the Ashkin-Teller model. We show that by selecting appropriate combinations of different types of damage ($q(q-1)$ in the Potts model and 12 in the Ashkin-Teller model), it is possible to find exact relations between damage and thermodynamic functions, like the magnetisation and the pair correlation function.

1. Introduction

The dynamical evolution of the Hamming distance between two microscopic configurations of a discrete statistical model is known as damage spreading.

In recent years, several efforts have been made, mainly for Ising models, to obtain a better understanding of this problem; In two [1, 2] and three [3, 4] dimensions as well as in the presence of a magnetic field [5], the Ising model has been exhaustively investigated by using different dynamics like heat-bath [3], Glauber [2, 4, 5] and Q2R [2].

Recently, Coniglio *et al* [1] have shown that for the Ising model [6] there are relations connecting the damage to thermal properties, and they used them to obtain good numerical estimates for the correlation function and the critical exponents associated with the clusters of correlated spins on a square lattice.

In this paper, we apply the approach introduced by Coniglio *et al* to the more general q-state Potts [7] and Ashkin-Teller [8] models, and we obtain exact relations between specific combinations of damages and macroscopic thermal quantities, like the pair correlation function and the magnetisation.

We investigate damage spreading by using two configurations (A and B) which evolve in time following the same dynamic rule (heat-bath, Glauber, etc), and we call a site damaged, at a given time, when the corresponding variables on this site are in different states in the two configurations, at that time.

In section 2 we discuss the Potts model, in section 3 the Ashkin-Teller model, and in section 4 we conclude.

2. The Potts model

The Potts model is defined by associating to each site of a lattice, a discrete variable

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σ, which can assume q values ($\sigma = 1, 2, 3, \ldots, q$). The Hamiltonian is given by:

$$\frac{\mathscr{H}}{k_B T} = -\sum_{\langle i,j \rangle} K_{ij}\delta(\sigma_i, \sigma_j) \tag{1}$$

where $K_{ij} \equiv J_{ij}/k_B T$ is the coupling constant between first-neighbouring sites and $\delta(\sigma_i, \sigma_j)$ is Kronecker's delta function.

If we consider two configurations of the model (A and B) there are at most $q(q-1)$ different types of damage that can occur on a site. We will, however, distinguish only two types of damage, namely

$$(1, \bar{1})_i \equiv \{\sigma_i(A) = 1;\ \sigma_i(B) \neq 1\} \qquad \text{and} \qquad (\bar{1}, 1)_i \equiv \{\sigma_i(A) \neq 1;\ \sigma_i(B) = 1\} \tag{2}$$

(the choice of state 1 is arbitrary).

If we start at the initial time ($t = 0$) with two configurations in thermal equilibrium, then the probabilities that the damages appear at the site i are given by:

$$d_i^{1\bar{1}} = [\delta(\sigma_i(A), 1)(1 - \delta(\sigma_i(B), 1))] \tag{3}$$

and

$$d_i^{\bar{1}1} = [(1 - \delta(\sigma_i(A), 1))\delta(\sigma_i(B), 1)] \tag{4}$$

where $[\ldots]$ denotes the time average over the trajectory in space phase.

We define the function Γ_i by:

$$\Gamma = d_i^{1\bar{1}} - d_i^{\bar{1}1}. \tag{5}$$

From (3)-(5) we have

$$\Gamma_i = [\delta(\sigma_i(A), 1)] - [\delta(\sigma_i(B), 1)]. \tag{6}$$

Using ergodicity and having all sites in both configurations free except the central site of configuration B, where we impose $\sigma_0(B) \neq 1$ for all values of $t \geq 0$, we have

$$[\delta(\sigma_i(A), 1)] = \langle \delta(\sigma_i, 1) \rangle \tag{7}$$

and

$$[\delta(\sigma_i(B), 1)] = \langle (1 - \delta(\sigma_0, 1))\delta(\sigma_i, 1) \rangle / \langle 1 - \delta(\sigma_0, 1) \rangle \tag{8}$$

($\langle \ldots \rangle$ is the thermal average).

From (6)-(8), we obtain the Γ_i function, with the above constraint at the origin,

$$\Gamma_i \equiv \Gamma_{0i} = \frac{C_{0i}}{2(1-m)} \tag{9}$$

where C_{0i} is the pair correlation function given by

$$C_{0i} = \langle \delta(\sigma_i, 1)\delta(\sigma_0, 1) \rangle - \langle \delta(\sigma_i, 1) \rangle \langle \delta(\sigma_0, 1) \rangle \tag{10}$$

and

$$m = \langle \delta(\sigma_0, 1) \rangle \tag{11}$$

(m is the order parameter, plus $1/q$, of the Potts model).

If we now fix simultaneously $\sigma_0(A) = 1$ and $\sigma_0(B) \neq 1$ for $t \geq 0$, it is easy to see that the Γ_i function (denoted as Γ'_{0i}) can be written as

$$\Gamma'_{0i} = \frac{C_{0i}}{1 - m^2}. \tag{12}$$

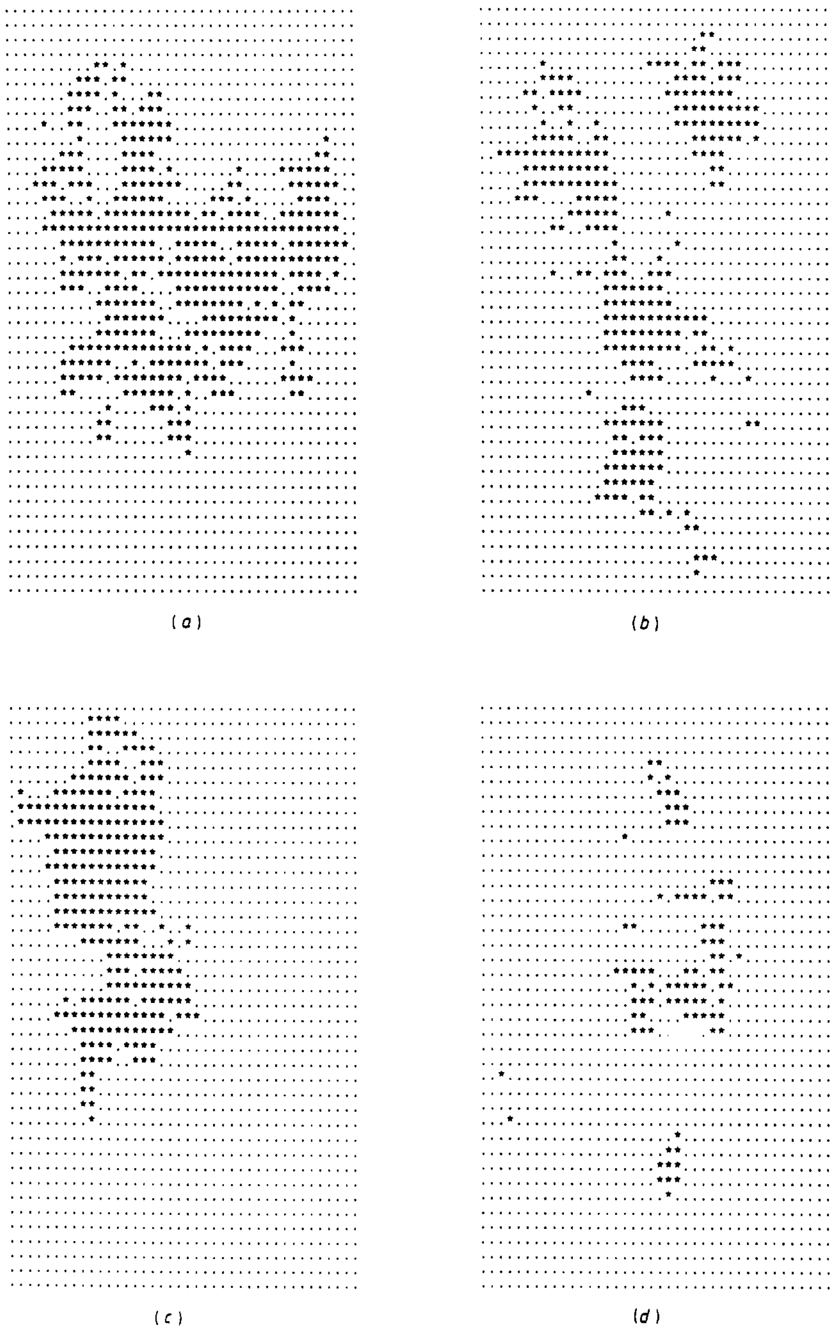

Figure 1. Typical large clusters of damaged sites $(1, \bar{1})$ of the Potts model at T_c, in a 40×40 square lattice. We have used the conditions: $\sigma_0(A) = 1$ and $\sigma_0(B) \neq 1$ for $t \geq 0$. (*a*) $q = 2$; (*b*) $q = 3$; (*c*) $q = 4$; (*d*) $q = 5$.

We see that (9)-(12) permit us to obtain the pair correlation function (C_{0i}) as well as information about the order parameter as functions of the combination of damages $(1, \bar{1})$ and $(\bar{1}, 1)$.

For the particular case $q = 2$ (Ising model) the expressions (9)-(12) recover, as expected, (6) and (7) of [1], if we set $\sigma_i = \pm 1$ and $\delta(\sigma_i, \sigma_j) = (1 + \sigma_i\sigma_j)/2$; it should be noticed that for $q > 2$, both damages $(1, \bar{1})$ and $(\bar{1}, 1)$ are always present for all dynamics (heat-bath, Glauber, etc) and initial conditions and, because of this, the Γ_{0i} and Γ'_{0i} functions cannot be identified with the total damage, as done in [1] for the Ising model ($q = 2$). Thus, it is necessary to calculate both damages to obtain the Γ_{0i} and Γ'_{0i} functions.

We have investigated the qualitative behaviour of clusters constituted by sites with a $(1, \bar{1})$ (or $(\bar{1}, 1)$) damage. For a square lattice at the critical temperature, we have noted that: (i) the sizes of these clusters strongly vary with time; (ii) for $q > 4$ (first-order phase transition) the clusters are very small and less connected than for the $q \leq 4$ case (second-order transition). In figure 1 we present some of the bigger clusters of $(1, \bar{1})$ damages sites, in a 40×40 lattice, for several values of q ($q = 2, \ldots, 5$), which appear over a period of 3600 time steps per site.

3. The Ashkin–Teller model

This model is constructed by associating to each site of a lattice two Ising variables σ, $\psi = \pm 1$, with interactions of two and four spins, described by the Hamiltonian:

$$\frac{\mathcal{H}}{k_B T} = -\sum_{\langle i,j \rangle} \{K_{ij}[\sigma_i\sigma_j + \psi_i\psi_j] + L_{ij}[\sigma_i\sigma_j\psi_i\psi_j]\} + \sum_i [h_i^\sigma \sigma_i + h_i^\psi \psi_i] \tag{13}$$

($[K_{ij} = J_{ij}^{(1)}/k_B T;\ L_{ij} = J_{ij}^{(2)}/k_B T]$ and $[h_i^\sigma = H_i^\sigma/k_B T;\ h_i^\psi = H_i^\psi/k_B T]$ are coupling constants and magnetic fields respectively).

For the present study, it is useful to define the binary variables: $\Pi_i^\sigma = ((1 + \sigma_i)/2); \Pi_i^\psi = ((1 + \psi_i)/2)$ and $\Pi_i^{\sigma\psi} = ((1 + \sigma_i\psi_i)/2)$, which take the role of the $\delta(\sigma_i, 1)$ in the Potts case. If we consider two configurations (A and B) of the model, we can see that, at each site, twelve different damages are possible. Their definitions and associated probabilities are given in table 1.

Table 1. Damages of the Ashkin–Teller model, with respective probabilities; [. . .] denotes time average over the space phase.

	$\sigma(A); \psi(A)$	$\sigma(B); \psi(B)$	Probabilities
1	$(+, +)$	$(+, -)$	$[\Pi^\sigma(A)\Pi^\psi(A)\Pi^\sigma(B)(1 - \Pi^\psi(B))]$
2	$(+, +)$	$(-, -)$	$[\Pi^\sigma(A)\Pi^\psi(A)(1 - \Pi^\sigma(B))(1 - \Pi^\psi(B))]$
3	$(+, +)$	$(-, +)$	$[\Pi^\sigma(A)\Pi^\psi(A)(1 - \Pi^\sigma(B))\Pi^\psi(B)]$
4	$(+, -)$	$(+, +)$	$[\Pi^\sigma(A)(1 - \Pi^\psi(A))\Pi^\sigma(B)\Pi^\psi(B)]$
5	$(+, -)$	$(-, +)$	$[\Pi^\sigma(A)(1 - \Pi^\psi(A))(1 - \Pi^\sigma(B))\Pi^\psi(B)]$
6	$(+, -)$	$(-, -)$	$[\Pi^\sigma(A)(1 - \Pi^\psi(A))(1 - \Pi^\sigma(B))(1 - \Pi^\psi(B))]$
7	$(-, +)$	$(+, +)$	$[(1 - \Pi^\sigma(A))\Pi^\psi(A)\Pi^\sigma(B)\Pi^\psi(B)]$
8	$(-, +)$	$(+, -)$	$[(1 - \Pi^\sigma(A)\Pi^\psi(A)\Pi^\sigma(B)(1 - \Pi^\psi(B)))]$
9	$(-, +)$	$(-, -)$	$[(1 - \Pi^\sigma(A)\Pi^\psi(A)(1 - \Pi^\sigma(B))(1 - \Pi^\psi(B)))]$
10	$(-, -)$	$(+, -)$	$[(1 - \Pi^\sigma(A)(1 - \Pi^\psi(A))\Pi^\sigma(B)(1 - \Pi^\psi(B)))]$
11	$(-, -)$	$(+, +)$	$[(1 - \Pi^\sigma(A))(1 - \Pi^\psi(A))\Pi^\sigma(B)\Pi^\psi(B)]$
12	$(-, -)$	$(-, +)$	$[(1 - \Pi^\sigma(A))(1 - \Pi^\psi(A))(1 - \Pi^\sigma(B))\Pi^\psi(B)]$

We will make a study similar to that of the Potts model; we start with two configurations in thermal equilibrium (with $h_i^\sigma = h_i^\psi = 0$) and on the central site we impose some restriction for all times ($t \geq 0$).

First, we select four types of combinations of damages:

$$(1,0)_i^{(1)} \equiv \{\Pi_i^\sigma(A) = 1; \Pi_i^\sigma(B) = 0\} \qquad (0,1)_i^{(1)} \equiv \{\Pi_i^\sigma(A) = 0; \Pi_i^\sigma(B) = 1\} \tag{14}$$

and

$$(1,0)_i^{(2)} \equiv \{\Pi_i^{\sigma\psi}(A) = 1; \Pi_i^{\sigma\psi}(B) = 0\} \qquad (0,1)_i^{(2)} \equiv \{\Pi_i^{\sigma\psi}(A) = 0; \Pi_i^{\sigma\psi}(B) = 1\}. \tag{15}$$

Denoting the respective probabilities by $D_i^{(1,0)}(1)$; $D_i^{(0,1)}(1)$, $D_i^{(1,0)}(2)$ and $D_i^{(0,1)}(2)$, we have

$$D_i^{(1,0)}(1) = d_i^{(2)} + d_i^{(3)} + d_i^{(5)} + d_i^{(6)} \tag{16}$$

$$D_i^{(0,1)}(1) = d_i^{(7)} + d_i^{(8)} + d_i^{(10)} + d_i^{(11)} \tag{17}$$

$$D_i^{(1,0)}(2) = d_i^{(1)} + d_i^{(3)} + d_i^{(10)} + d_i^{(12)} \tag{18}$$

$$D_i^{(0,1)}(2) = d_i^{(4)} + d_i^{(6)} + d_i^{(7)} + d_i^{(9)} \tag{19}$$

where $d_i^{(j)} (j = 1, 12)$ are defined in table 1.

Now, we define two functions $\Gamma_i^{(1)}$ and $\Gamma_i^{(2)}$ by

$$\Gamma_i^{(1)} = D_i^{(1,0)}(1) - D_i^{(0,1)}(1) \tag{20}$$

and

$$\Gamma_i^{(2)} = D_i^{(1,0)}(2) - D_i^{(0,1)}(2). \tag{21}$$

Using (16)-(21) and the values of table 1, we obtain

$$\Gamma_i^{(1)} = [\Pi_i^\sigma(A)] - [\Pi_i^\sigma(B)] \tag{22}$$

and

$$\Gamma_i^{(2)} = [\Pi_i^{\sigma\psi}(A)] - [\Pi_i^{\sigma\psi}(B)] \tag{23}$$

(we can analogously define a similar quantity for the Π_i^ψ variable).

Finally, we impose appropriate boundary conditions to obtain relations between the Γ_i and thermal functions.

3.1. One-spin function

If one imposes the boundary conditions: $\sigma_0(B) = -1 (t \geq 0)$ one has:

$$[\Pi_i^\sigma(A)] = \langle \Pi_i^\sigma \rangle \qquad \text{and} \qquad [\Pi_i^\sigma(B)] = \frac{\langle (1 - \Pi_0^\sigma)\Pi_i^\sigma \rangle}{\langle 1 - \Pi_o^\sigma \rangle} \tag{24}$$

then

$$\Gamma_i^{(1)} \equiv \Gamma_{0i}^{(1)} = \frac{C_{0i}^{(1)}}{2(1-m)} \tag{25}$$

where $C_{0i}^{(1)} = \langle \sigma_i \sigma_0 \rangle - \langle \sigma_i \rangle \langle \sigma_0 \rangle$ is the pair correlation function and $m = \langle \sigma_0 \rangle$ is the one-spin magnetisation.

By imposing the boundary condition: $\sigma_0(A) = 1$ and $\sigma_0(B) = -1$ ($t \geq 0$) we can easily prove that:

$$\Gamma_i^{(1)} \equiv \Gamma_{0i}^{\prime(1)} = \frac{C_{0i}^{(1)}}{(1-m^2)}. \tag{26}$$

3.2. Two-spin functions

To obtain the two-spin thermal functions, we can choose the following boundary condition: $\sigma_0(B) = -\psi_0(B)$ $(t \geq 0)$. This implies that

$$[\Pi_i^{\sigma\psi}(A)] = \langle \Pi_i^{\sigma\psi} \rangle \tag{27}$$

and

$$[\Pi_i^{\sigma\psi}(B)] = \frac{\langle (1 - \Pi_0^{\sigma\psi}) \Pi_i^{\sigma\psi} \rangle}{\langle 1 - \Pi_0^{\sigma\psi} \rangle}. \tag{28}$$

Using (27) and (28) in (23) one has

$$\Gamma_i^{(2)} \equiv \Gamma_{0i}^{(2)} = \frac{C_{0i}^{(2)}}{2(1-M)} \tag{29}$$

where $C_{0i}^{(2)} = \langle \sigma_i \psi_i \sigma_0 \psi_0 \rangle - \langle \sigma_i \psi_i \rangle \langle \sigma_0 \psi_0 \rangle$ and $M = \langle \sigma_0 \psi_0 \rangle$.

Finally, if one imposes the boundary conditions $\sigma_0(A) = \psi_0(A)$ and $\sigma_0(B) = -\psi_0(B)$ $(t \geq 0)$, we can see that:

$$\Gamma_i^{(2)} \equiv \Gamma_{0i}^{\prime(2)} = \frac{C_{0i}^{(2)}}{1 - M^2}. \tag{30}$$

Consequently, the pair correlation functions $(C_{0i}^{(1)}, C_{0i}^{(2)})$ and magnetisations (m, M) can be calculated, by (25), (26) and (29), (30), as functions of $\Gamma_i^{(1)}$ and $\Gamma_i^{(2)}$.

We have also examined the clusters of sites with a $(1,0)^{(1)}$ damage, as well as the clusters of sites with a $(1,0)^{(2)}$ damage, in a 40×40 square lattice, on Baxter's critical line [9] (on which the critical exponents change continuously). The susceptibilities ($\chi^{(1)}$ and $\chi^{(2)}$), associated with the pair correlation functions ($\chi^{(j)} = \Sigma_i C_{0i}^{(j)}; j = 1, 2$), are expected to obey the scaling laws [1] $\chi^{(j)} \propto L^{\gamma(j)/\nu(j)}$, where L is the size of the system and $\gamma(j)$ and $\nu(j)$ are, respectively, the critical exponents associated with the susceptibility and the correlation length for the one $(j = 1)$ and two spins $(j = 2)$. On Baxter's line these exponents are known [9] (for one spin $\gamma^{(1)}/\nu^{(1)} = 7/4$ and for two spins $3/2 \leq \gamma^{(2)}/\nu^{(2)} \leq 7/4$); however, we have noted that all clusters of $(1,0)^{(1)}$ and $(1,0)^{(2)}$ damaged sites (observed for 20 000 time steps per site) are very small (≈ 40) so that it is difficult to draw satisfactory conclusions from the numerical data.

3.3. Another approach to obtain the magnetisations

To calculate the magnetisations m and M, it is more useful to consider the configurations A and B in the presence of magnetic fields (h_i^σ and h_i^ψ), without fixing any spin.

We start by setting on all sites of the lattice:

$$\sigma_i(A) = -\sigma_i(B);\ \psi_i(A) = \psi_i(B);\ h_i^\sigma(A) = -h_i^\sigma(B) = h_i^\psi(A) = h_i^\psi(B) = h.$$

Now considering the Ashkin–Teller model in the presence of a magnetic field $h_i^\sigma = h_i^\psi = h$, and using the invariance of the averages under $\sigma_i \to -\sigma_i$; $\psi_i \to \psi_i$; $h_i^\sigma \to -h_i^\sigma$; $h_i^\psi \to h_i^\psi$, one has

$$[\Pi_i^\sigma(A)] = \langle \Pi_i^\sigma \rangle \tag{31}$$

$$[\Pi_i^\sigma(B)] = \langle 1 - \Pi_i^\sigma \rangle \tag{32}$$

$$[\Pi_i^{\sigma\psi}(A)] = \langle \Pi_i^{\sigma\psi} \rangle \tag{33}$$

$$[\Pi_i^{\sigma\psi}(B)] = \langle 1 - \Pi_i^{\sigma\psi} \rangle. \tag{34}$$

Then using (22) and (23) we obtain

$$\Gamma_i^{(1)} = \langle \sigma_i \rangle = m \tag{35}$$

and

$$\Gamma_i^{(2)} = \langle \sigma_i \psi_i \rangle = M. \tag{36}$$

4. Conclusions

In this paper, we have generalised the results, already known for the Ising model, relating the damage spreading to thermal properties to the case of the Potts and the Ashkin–Teller models.

The generality of the models studied here, makes the analysis of damages more difficult than in the Ising case. In general, there are more damages ($q(q-1)$ in Potts and 12 in the Ashkin–Teller case), and we have not been able to choose a boundary condition and a particular dynamics to eliminate a large fraction of them, as was done in the Ising case.

However, the exact relations developed here, allow us to obtain numerically accurate values of thermal properties (like the pair correlation function, the susceptibility and the magentisation) from the knowledge of selected combinations of damages, using any dynamics (heat-bath, Glauber, etc). In particular, these calculations can be very useful to obtain a better understanding of these models on three-dimensional lattices, where no exact solution is known at present.

Another interesting result obtained here, was the qualitative aspect of the clusters of damaged sites in the Potts case, at the critical temperature on a square lattice; we noted that when q (number of states) is greater than 4, the clusters become very small and less connected (as compared with the ones for $q \leqslant 4$). This result is related to the change of the order of the phase transition at $q = 4$ (second-order ($q \leqslant 4$) to first-order ($q > 4$)). In fact, the correlation function, that is related to the damages, is, as we know, long-ranged in a continuous transition, becoming short-ranged when the model exhibits a first-order phase transition.

It should be interesting to investigate these clusters in three dimensions, to obtain, in particular, information about the type of phase transitions exhibited by the antiferromagnetic Potts as well as the Ashkin–Teller models.

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